Reservation Planning for Elective Surgery Under Uncertain Demand for Emergency Surgery

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Presented by Eric Webb

At the beginning of the day, there are $n$ requests for elective surgery.
The scheduler decides how many of them, $m$, to admit today.
There is a random, unknown duration of emergency surgeries throughout the day.
These emergency cases take priority over elective cases.
Overloading operating room capacity leads to high costs of overtime pay and/or transporting patients to nearby hospitals.
The Problem

- How do you effectively schedule elective surgeries in an operating room?
- While the number of elective surgery requests is known, the duration of each surgery is unknown.
- The number and duration of emergency surgeries is unknown.
- Hospitals make revenue from elective surgeries.
- Hospitals lose money by exceeding operating room capacity and having to pay overtime.
- We also consider a penalty to delaying elective surgeries.
Motivation

- A cutoff policy is a policy in which every elective surgery is admitted, up to a cutoff number $M$. After $M$ elective surgeries are admitted, no others are admitted that day.
- $M$ is based on hospital capacity and expected duration of emergency cases.
- Most hospitals use a cutoff policy.
- We want to see if there is a better policy for admitting elective surgery patients.
Expected Revenue in One Day

- Let $X_j$ be the number of new requests for surgery on the $j^{th}$ day.
- Let $n$ be the total number of current requests for surgery, some of which are new and some of which are past requests which have not been fulfilled.
- Let $m$ be the number of elective surgeries to admit today.
- Let $\pi$ be the expected revenue from each elective surgery.

- Present day expected revenue: $\pi m$
Expected Overtime Costs in One Day

- Let $Z_i$ be the unknown duration of the $i^{th}$ elective surgery of the day ($i \in \{1, 2, ..., m\}$).

- Let $S_m = \sum_{i=1}^{m} Z_i$ be the total duration of elective surgeries today.

- Let $Y$ be the random duration of emergency surgeries today.

- Let $T$ be the duration the operating room can be used before overtime pay begins. This value is known.

- Let $c$ be the penalty, per unit time, of exceeding the day’s capacity, $T$.

- Present day expected overtime cost: $c \mathbb{E}[(Y + S_m - T)^+]$
Expected Delay Penalty in One Day

- The waiting patient and/or society often incurs a cost when individuals are fully or partially unable to function normally.
- Let $p$ be the average daily penalty for the postponement of each elective case by one day.
- Present day expected penalty from postponement: $p(n - m)$
Present day expected profit function

- Out of a pool of \( n \) possible elective surgeries, \( m \) are performed.
- Expected Profit = Elective Surgery Revenue - Expected Overtime Costs - Postponement Penalties

\[
g(m, n) = \pi m - c\mathbb{E}[(Y + S_m - T)^+] - p(n - m)
\]
Concavity of Present Day Profit

Theorem: The function $g(m, n)$ is jointly concave in $m$ and $n$ for every $T \geq 0$. Furthermore, if $\pi + p < cE[Z_1]$, then $g(m, n)$ is bounded from above.

Interpretation:

- We expect the cost of overtime to be greater than the sum of the profit of the extra surgery and the savings of not postponing the surgery.

- There is a maximum on how much profit can be made in this case. The maximum profit is dependent on the operating room capacity, $T$. 
Discounting Future Profit

- Let $f_i(n)$ be the maximal expected discounted profit with $i$ days remaining if there are $n$ outstanding elective cases at the beginning of that day.

- Dynamic programming recursion for determining today’s allotment:

$$f_i(n) = \max_{m \leq n} (g(m, n) + \alpha \mathbb{E}[f_{i-1}(n - m + X)])$$

where $g(m, n)$ is the present day expected profit, $\alpha$ is the daily discount factor ($0 < \alpha < 1$), and $X$ is the number of new arrivals the next day.

- $f_0(n) = 0 \ \forall \ n$

- We will apply the backward recursion an “infinite” number of times to model the reality of an ongoing operating room.
Concavity of $f_i(n)$

Theorem: $f_i(n)$ is concave in $n \forall i$. If $p = 0$, then $f_i(n)$ is also increasing in $n$.

Interpretation:

- Like our present-day profit examined in the previous theorem, our discounted future profit is also concave in $n$.
- If there is no penalty to postponing surgeries, then our profit cannot be diminished by having more possible surgeries to schedule.
Discounted Profit of an Ongoing Operating Room

Theorem: If $\pi + p < c\mathbb{E}[Z_1]$, then there exists a concave function $f$ such that $\lim_{i \to \infty} f_i = f$, and which satisfies

$$f(n) = \max_{m \leq n} \left( g(m, n) + \alpha \mathbb{E}[f(n - m + X)] \right)$$

Interpretation: The use of most operating rooms is expected to continue indefinitely. So, without an ending date in sight, this theorem tells us that there is a concave discounted profit function to use to find the optimal policy.
Characterizing the Optimal Number to Admit

Let \( m(n) \) be the value of \( m \) that maximizes \( f(n) \), the maximal expected discounted profit.

\( m(n) \) is non-decreasing in \( n \)

- Numerical example:
  If you would admit 4 surgeries out of a possible 5 surgeries, then you would admit 4 or more out of a possible 6 surgeries.

For every \( n \geq 0 \) and \( d \geq 0 \), \( m(n + d) - m(n) \leq d \)

- Numerical example:
  If you would admit 5 elective surgeries out of a possible 10, then you would admit no more than 7 elective surgeries out of a possible 12.

\( m \) can be found through Value Iteration.
Effect of Arrival Rate

Units are minutes
\[ T = 960, \ Y \sim \text{Norm}(400, 640), \ Z_i \sim \text{Norm}(60, 100), \]
\[ \pi = \$600, \ c = \$15, \ \alpha = .99, \ \rho = \$0, \]
Arrivals are Poisson with rate listed
Effect of Overtime Cost

Units are minutes

\[ T = 960, \ Y \sim \text{Norm}(400, 640), \ Z_i \sim \text{Norm}(60, 100), \]

\[ X \sim \text{Poisson}(10), \ \pi = $600, \ \alpha = .99, \ p = $0 \]
Effect of Postponement Penalty

Units are minutes
\[ T = 960, \; Y \sim \text{Norm}(400, 640), \; Z_i \sim \text{Norm}(60, 100), \]
\[ X \sim \text{Poisson}(10), \; \pi = \$600, \; c = \$15, \; \alpha = .99, \]
Comparison to Cutoff Policy

- It is possible to re-formulate this system to find the best cutoff policy number, $M$.
  - Add the constraint that $m(n) = \min(n, M)$, where $M$ is the cutoff number.
  - Find the cutoff number that maximizes $f(n)$

- We will compare the optimal policy found in this model to the best cutoff policy to see if there are significant differences.
Avg Arrival Rate, $E(X) = 10$

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<th>Cutoff # Policy</th>
<th>Optimal Policy</th>
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Conclusions

- It is possible to find the optimal policy for making reservations for elective surgery in the face of uncertain demand for emergency surgery.

- The optimal policy is not one of cutoff number, but the relative loss in profit from using the best cutoff number policy is small.

- Finding the optimal policy suggests the best cutoff number.